Note Title 3/13/2008

Knowing E_0 & ψ_0, we can find all ψ_n's and E_n's:

$$\psi_1 = A_1 \hat{a}^\dagger \psi_0 \rightarrow E_1 = E_0 + \hbar\omega = \frac{\hbar\omega}{2} + \hbar\omega = \hbar\omega(\tfrac{1}{2}+1)$$

$$\psi_2 = A_2 \hat{a}^\dagger \psi_1 \rightarrow E_2 = E_1 + \hbar\omega = \hbar\omega(\tfrac{1}{2}+2)$$

$$\vdots$$

$$\psi_n = A_n \hat{a}^\dagger \psi_{n-1} \rightarrow E_n = \hbar\omega(\tfrac{1}{2}+n)$$

$$\boxed{E_n = \hbar\omega(n+\tfrac{1}{2})} \qquad n = 0, 1, 2, \ldots$$

Compare with Schrodinger equation:

$$H|n\rangle = E_n|n\rangle$$

$$\hbar\omega(a^\dagger a + \tfrac{1}{2})|n\rangle = \hbar\omega(n+\tfrac{1}{2})|n\rangle$$

$$\hbar\omega\, a^\dagger a|n\rangle + \cancel{\frac{\hbar\omega}{2}|n\rangle} = \hbar\omega\, n|n\rangle + \cancel{\frac{\hbar\omega}{2}|n\rangle}$$

$$\Rightarrow \boxed{a^\dagger a|n\rangle = n|n\rangle}$$

$a^\dagger a$ is often called "number" operator:

$$N \equiv a^\dagger a$$

$$N|n\rangle = n|n\rangle$$

$\hat{a}$ annihilation (or lowering): $\psi_{n-1} = A_{n-1}\hat{a}\,\psi_n$

$\hat{a}^\dagger$ Creation (or raising): $\psi_{n+1} = A_{n+1}\hat{a}^\dagger\,\psi_n$

A_{n-1} and A_{n+1} are some constants that are determined by normalization. Let's assume ψ_n is normalized:

$$\int \psi_n^* \psi_n \, dx = \langle \psi_n | \psi_n \rangle = 1 \quad \text{or simply write: } \langle n | n \rangle = 1$$

Because the state $|n+1\rangle$ must be also normalized:

$$\langle n+1 | n+1 \rangle = 1$$

$$\left.\begin{aligned} |n+1\rangle &= A_{n+1} \hat{a}^+ |n\rangle \\ \langle n+1| &= A_{n+1}^* \langle n | \hat{a} \end{aligned}\right\} \langle n+1 | n+1 \rangle = |A_{n+1}|^2 \langle n | \underbrace{\hat{a}\hat{a}^+}_{a^+a+1} |n\rangle = 1$$

because $[a, a^+] = 1$

$$\Rightarrow \quad |A_{n+1}|^2 \langle n | a^+a + 1 | n \rangle = 1$$

$$|A_{n+1}|^2 \langle n | \underbrace{a^+a}_{N} | n \rangle + |A_{n+1}|^2 \underbrace{\langle n | n \rangle}_{=1} = 1$$

$$|A_{n+1}|^2 \, n \underbrace{\langle n | n \rangle}_{1} + |A_{n+1}|^2 = 1$$

$$|A_{n+1}|^2 (n+1) = 1 \Rightarrow \quad A_{n+1} = \frac{1}{(n+1)^{1/2}} \quad \Rightarrow$$

$$\psi_{n+1} = A_{n+1} \hat{a}^+ \psi_n \rightarrow |n+1\rangle = \frac{1}{(n+1)^{1/2}} \hat{a}^+ |n\rangle$$

Similarly we can show: $|n-1\rangle = \frac{1}{n^{1/2}} \hat{a} |n\rangle$

or:

$$\boxed{\begin{aligned}\hat{a}^{+}|n\rangle &= \sqrt{n+1}\ |n+1\rangle \\ \hat{a}\,|n\rangle &= \sqrt{n}\ |n-1\rangle\end{aligned}}$$

Suppose we normalize the ground state: $\langle 0|0\rangle = 1$

using $|n+1\rangle = \frac{1}{(n+1)^{1/2}}\, a^{+}|n\rangle \Rightarrow$

$$|1\rangle = \frac{1}{1^{1/2}}\, a^{+}|0\rangle$$

$$|2\rangle = \frac{1}{2^{1/2}}\, a^{+}|1\rangle = \frac{(a^{+})^{2}}{2^{1/2}\,1^{1/2}}|0\rangle$$

$$\vdots$$

$$|n\rangle = \frac{(a^{+})^{n}}{(n!)^{1/2}}|0\rangle$$

Matrix element

$$\langle n|\underset{\substack{\downarrow\\ \text{operator}}}{\hat{\mathcal{L}}}|m\rangle = \int \psi_n \hat{\mathcal{L}} \psi_m\, d^3r \equiv \text{Matrix element}$$

For example $\langle n|a^{+}|m\rangle$ is a matrix element.

Ex/ $\langle n|a^{+}|n\rangle = \sqrt{n+1}\ \underbrace{\langle n|n+1\rangle}_{0} = 0$

$$\langle n|a^{+}|n-1\rangle = \sqrt{n}\ \underbrace{\langle n|n\rangle}_{1} = \sqrt{n}$$

In general: $\langle n|a^{+}|m\rangle = \sqrt{n}\ \delta_{n,m+1} = \begin{cases}\sqrt{n} & n = m+1 \\ 0 & n \neq m+1\end{cases}$

$$\langle n|a|m\rangle = \sqrt{n}\ \delta_{n,m-1}$$